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GRADED LIE ALGEBRAS OF MAXIMAL CLASS. V

BY

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ABSTRACT

We describe the isomorphism classes of certain infinite-dimensional graded Lie algebras of maximal class, generated by an element of weight one and an element of weight two, over fields of even characteristic.

1. Introduction

Let M be a Lie algebra. Suppose M is nilpotent of nilpotency class c, so that c is the smallest number such that the Lie power M^{c+1} vanishes. If M has finite dimension $n \geq 2$, it is well-known that $c \leq n-1$. When $c = n-1$, M is said to be a Lie algebra of maximal class. Equivalently, a Lie algebra M of dimension n is of maximal class when the i-th Lie power M^i has codimension i in M, for

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$2 \leq i \leq n$. An infinite-dimensional Lie algebra M is said to be of maximal class when M^i has codimension i for all $i \geq 2$, and M is residually nilpotent, that is, $\bigcap_i M^i = \{0\}$.

Infinite-dimensional graded Lie algebras of maximal class which can be generated by two elements of weight one have been classified in a series of papers [3, 6, 4, 11]. (We call these **algebras of type 1**.) This is a problem only in positive characteristic, as it is well-known that there is only one such algebra in characteristic zero. Over any field $\mathbf{F}$ of positive characteristic, there are $|\mathbf{F}|^{\aleph_0}$ algebras of type 1.

There are other types of graded Lie algebras of maximal class: in [13] those infinite-dimensional graded Lie algebras of maximal class

$$L = \bigoplus_{i \geq 1} L_i$$

have been studied, in which $\dim(L_i) = 1$ for all $i \geq 1$, so that they are generated by an element of weight one and an element of weight two. (We call these **algebras of type 2**.) In [13] it has been shown that in characteristic zero there are three (isomorphism types of) algebras of type 2. In [5] the same problem has been solved over fields of odd characteristic. In this paper we cover the case of fields of characteristic two.

In [5] we described a method for obtaining algebras of type 2 as maximal subalgebras of *uncovered* algebras of type 1 (see the next section for the details), and showed that over a field of characteristic $p > 2$ the algebras of type 2 consist of

- algebras arising as maximal subalgebras of algebras of type 1,
- a certain soluble algebra,
- one further family of soluble algebras,
- in the case $p = 3$, one additional family of soluble algebras.

Despite the additional complication with the prime 3, it turns out that the situation in characteristic 2 can be described in a more uniform way, in terms of the *sequences of two-step centralizers* of the involved algebras. (We define sequences of two-step centralizers below.)

The plan of this paper is the following. In Section 2 we set up some preliminaries, and state our main result (Theorem 2.4). In Section 3 we show that the algebras we study fall into two main categories. The first one is dealt with in Section 4, the second one in Section 5.

2. Preliminaries

From now on, **F** will be a fixed field of positive characteristic p. Later we will take $p = 2$. All Lie algebras are taken over **F**.

Suppose $N = \bigoplus_{i \geq 1} N_i$ is an algebra of type 1. We say that N is **uncovered** when there is an element $e_1 \in N_1$ such that $[N_i e_1] = N_{i+1}$ for $i \geq 1$. (It is shown in [3] that over any countable field there are algebras of type 1 that are not uncovered.) Suppose that N is uncovered. Choose $0 \neq e_2 \in N_2$, and define recursively $e_{i+1} = [e_i e_1]$, for $i \geq 2$. Since N is uncovered, e_i spans N_i, for $i \geq 2$.

We now give a definition of **constituents** for uncovered algebras of type 1, suitable for the purposes of this paper; this is a non-normalized version of that of [4]. Choose $y \in N_1 \setminus \langle e_1 \rangle$. For each $i \geq 2$ we have $[e_i y] = \beta_i e_{i+1}$, for some $\beta_i \in \mathbf{F}$. We call $(\beta_i)_{i \geq 2}$ the **sequence of two-step centralizers** of N. (This definition is motivated by the corresponding one for p-groups of maximal class: see [1] or [7, III.14].)

Suppose $\beta_2 = \beta_3 = \cdots = \beta_{n-1} \neq \beta_n$ for some n. Then we refer to the sequence $\beta_2, \beta_3, \ldots, \beta_{n-1}, \beta_n$ as the **first constituent**. The length of this constituent is defined to be n. (This convention allows for simpler statements of several results.) We now define the other constituents recursively. Suppose $\beta_i, \ldots, \beta_j$ is a constituent we have already defined, and that for some $m > 1$ we have

$$\beta_{j+1} = \cdots = \beta_{j+m-1} \neq \beta_{j+m}.$$

Then $\beta_{j+1}, \ldots, \beta_{j+m}$ is defined to be a constituent of length m.

Suppose $N = \bigoplus_{i \geq 1} N_i$ is an uncovered algebra of type 1, and suppose $e_1 \in N_1$ has the property that $[N_i e_1] = N_{i+1}$ for all $i \geq 2$. Write $L_1 = \langle e_1 \rangle$, and $L_i = N_i$, for $i > 1$. It is clear that the maximal, graded subalgebra $\bigoplus_{i \geq 1} L_i$ of N is an algebra of type 2.

Now let

$$L = \bigoplus_{i \geq 1} L_i$$

be an algebra of type 2. As for the case of algebras of type 1 above, choose $0 \neq e_2 \in L_2$, and define recursively $e_{i+1} = [e_i e_1]$, for $i \geq 2$. It follows that $L_i = \langle e_i \rangle$ for all $i \geq 1$. There are elements $\alpha_i \in \mathbf{F}$, for $i \geq 3$, such that $[e_i e_2] = \alpha_i e_{i+2}$. We refer to the sequence $(\alpha_i)_{i \geq 3}$ as the **sequence of two-step centralizers of** L. The sequence of two-step centralizers completely determines the multiplication table of the algebra L. This follows from an application of the generalized Jacobi identity:

$$\begin{aligned}[e_k e_j] &= [e_k[e_2 \underbrace{e_1 \dots e_1}_{j-2}]] \\ &= \sum_{i=0}^{j-2} (-1)^i \binom{j-2}{i} [e_k \underbrace{e_1 \dots e_1}_{i} e_2 \underbrace{e_1 \dots e_1}_{j-2-i}] \\ &= \left(\sum_{i=0}^{j-2} (-1)^i \binom{j-2}{i} \alpha_{k+i} \right) e_{k+j}.\end{aligned}$$

Remark: In the above, we may replace e_2 by a multiple $e_2' = \beta e_2$, for some $0 \neq \beta \in \mathbf{F}$, and accordingly $e_3' = [e_2' e_1]$ etc., so that $e_i' = \beta e_i$. Clearly $[e_i' e_2'] = \beta \alpha_i e_{i+2}'$.

It follows that one can *scale* the sequence of two-step centralizers of an algebra of type 2 by a non-zero factor, without affecting the isomorphism type of the underlying algebra.

Furthermore, it is easy to see that two sequences of two-step centralizers determine isomorphic algebras if and only if one can be obtained from the other by scaling. So we can classify algebras of type 2 by classifying possible sequences of two-step centralizers.

We now record a simple fact, which we make repeated use of in this article. In its proof, and in the rest of the paper, we will use the notation

$$[e_2 e_1^i] = [e_2 \underbrace{e_1 \dots e_1}_{i}]. \tag{2.1}$$

2.1. LEMMA: *If $m \geq 2$ then α_{2m} can be expressed as a linear combination with integer coefficients of $\alpha_{m+1}, \alpha_{m+2}, \dots, \alpha_{2m-1}$. Namely, we have*

$$\alpha_{2m} = \sum_{i=0}^{m-2} (-1)^{m+i} \binom{m-1}{i} \alpha_{m+1+i}. \tag{2.2}$$

Proof: Consider the relation $[e_{m+1} e_{m+1}] = 0$. Since $e_{m+1} = [e_2 e_1^{m-1}]$, we have

$$\begin{aligned}0 = [e_{m+1}[e_2 e_1^{m-1}]] &= \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} [e_{m+1} e_1^i e_2 e_1^{m-1-i}] \\ &= \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} \alpha_{m+1+i} e_{2m+2}.\end{aligned}$$

This yields (2.2). ∎

Let L be an algebra of type 2 that is a maximal subalgebra of an algebra N of type 1, and let $e_1, y \in N$ be as above. There is a simple relation between the sequences of two-step centralizers of N and L. Taking $e_2 = [e_1 y]$, we have

$$\alpha_i e_{i+2} = [e_i e_2] = [e_i[e_1 y]] = [e_i e_1 y] - [e_i y e_1] = (\beta_{i+1} - \beta_i) e_{i+2},$$

so that $\alpha_i = \beta_{i+1} - \beta_i$.

This suggests the following definition for constituents in algebras of type 2; this is a slight variation on that of [5, Section 2], that is better suited to the case of fields of characteristic 2. If $\alpha_3 \neq 0$, then we can assume by scaling that $\alpha_3 = 1$, and it follows from Lemma 2.1 that we also have $\alpha_4 = 1$. We refer to the leading 11 of the sequence of two-step centralizers in this case as the first constituent, which we regard of length 4. If $\alpha_3 = \ldots = \alpha_{n-2} = 0$, but $\alpha_{n-1} \neq 0$, we refer to the leading $0 \ldots 0 \alpha_{n-1} \alpha_n$ of the sequence of two-step centralizers as the first constituent, which we regard of length n. (As in the case of algebras of type 1, the case of the first constituent has to be dealt with separately to fit the scheme.) Proceeding recursively, if $\alpha_i, \ldots, \alpha_j$ is a constituent we have already defined, and if we have $\alpha_{j+1} = \ldots = \alpha_{j+m-2} = 0$, but $\alpha_{j+m-1} \neq 0$, then we define $\alpha_{j+1} \ldots \alpha_{j+m}$ to be a constituent of length m.

It follows from the above and Lemma 2.5 below that if L is an algebra of type 2 which is a maximal subalgebra of an algebra N of type 1, then all constituents of L will be of the form $0, 0, \ldots, \lambda, -\lambda$, for some $\lambda \neq 0$. Also, the lengths of the constituents of N and L will be the same. We refer to a constituent of the form

$$0, 0, \ldots, 0, \lambda, -\lambda$$

of an algebra of type 2 as an **ordinary constituent ending in** λ. In [5, Section 2] we have proved that the converse holds:

2.2. PROPOSITION: *If all constituents of the algebra L of type 2 are ordinary, then L is a maximal subalgebra of an algebra of type 1.*

It is sometimes convenient to describe an algebra, or part of it, through the sequence of lengths of its constituents. For instance, in Section 5 we will deal with algebras of type 2 in characteristic 2 whose sequence of two-step centralizers begins

$$\underbrace{1 \ldots 1}_{2(q-1)} 011,$$

for some q. According to the definitions above, all constituents here are ordinary ending in 1, and the sequence of their lengths is $4 2^{q-2} 3$, where 2^{q-2} denotes repetition $q-2$ times of the length 2.

The next trick plays an important role in the formulation of our main result; for it we need to take the characteristic p of the underlying field $\mathbf{F}$ to be 2. Regard the algebra L of type 2 as embedded in an associative algebra A. In the rest of this section only, powers of an element of a Lie algebra refer to powers in such an associative algebra. In the rest of the paper we will use powers to denote repeated commutators, such as in (2.1).

Take $\gamma \in \mathbf{F}$, and consider the Lie subalgebra $L(\gamma)$ of A generated by e_1 and $e_2' = e_2 + \gamma e_1^2$. We have

$$\begin{aligned}[e_i e_2'] &= [e_i, e_2 + \gamma e_1^2] \\ &= [e_i e_2] + \gamma[e_i e_1^2] = \alpha_i e_{i+2} + \gamma e_{i+2} \\ &= (\alpha_i + \gamma) e_{i+2}.\end{aligned}$$

Here we have used the fact (see [8, V.7]) that in characteristic 2 we have

$$e_{i+2} = [e_i e_1 e_1] = e_i e_1^2 - 2e_1 e_i e_1 + e_1^2 e_i = [e_i e_1^2]. \tag{2.3}$$

Also, $[e_2' e_1] = [e_2 + \gamma e_1^2, e_1] = e_3$. It follows that

$$L(\gamma) = L_1 \oplus \langle e_2' \rangle \oplus \bigoplus_{i \geq 3} L_i$$

is an algebra of type 2, with sequence of two-step centralizers $\alpha_i' = \alpha_i + \gamma$.

We have obtained

2.3. Lemma: *Suppose the characteristic of the underlying field $\mathbf{F}$ is 2.*

If we add a constant γ to all the terms of the sequence of two-step centralizers of an algebra of type 2, we obtain the sequence of two-step centralizers of another algebra of type 2.

We are now able to state our main result.

2.4. Theorem: *Let N be an uncovered algebra of type 1 over a field of characteristic 2, with sequence of two-step centralizers $(\beta_i)_{i \geq 2}$*

Then for every constant γ the sequence $(\beta_{i+1} + \beta_i + \gamma)_{i \geq 3}$ is the sequence of two-step centralizers of an algebra of type 2.

Conversely, the sequence of two-step centralizers of every algebra of type 2 over a field of characteristic 2 arises in this way.

Note that the possible sequences of two-step centralizers of algebras of type 1 are classified in [CMN, Cla, CN, Jur], so that Theorem 2.4 provides a complete classification of possible sequences of two-step centralizers of algebras of type 2.

This in turn, as we observed above, provides a classification of algebras of type 2.

We now recall some facts about algebras of type 1 that will be useful in the sequel. They are valid in arbitrary characteristic $p > 0$.

2.5. LEMMA/DEFINITION (Propositions 3.1, 3.2, and Definition 3.3 of [4]): *Let N be an infinite-dimensional algebra of type 1 in characteristic $p > 0$.*

All the elements of a constituent, except the last one, coincide with the first two-step centralizer β_2.

The first constituent of N has length $2p^h$, for some $h \geq 1$. We call $q = p^h$ the **parameter** *of N.*

The other constituents can have length $2q$, or $2q - p^k$, for some $0 \leq k \leq h$.

2.6. LEMMA (Step 4.6 of [4, 11]): *Let N be an infinite-dimensional algebra of type 1 and parameter q. Suppose N has a constituent which has length neither $2q$ nor q. Then N has exactly two distinct two-step centralizers.*

2.7. LEMMA (Lemma 3.1 of [4]; see also the Lemma of [2]): *Let N be an infinite-dimensional algebra of type 1 and parameter q. Let $(\beta_i)_{i\geq 2}$ be the sequence of two-step centralizers of N. Let β be a two-step centralizer other than the first two in order of occurrence, that is, $\beta \neq \beta_2, \beta_{2q}$.*

Then every instance of β occurs at the end of a constituent of length q, and it is followed by another constituent of length q.

2.8. LEMMA (Step 4.10 of [4, 11]): *Let N be an infinite-dimensional algebra of type 1 and parameter q. Let β be a two-step centralizer. Then β first occurs in the sequence of two-step centralizers as $\beta = \beta_{2p^n}$, for some n.*

Finally, we mention *deflation* in characteristic 2. Let $L = \bigoplus_{i\geq 1} L_i$ be an algebra of type 2, with $L_i = \langle e_i \rangle$, and $[e_i e_1] = e_{i+1}$, for $i \geq 2$. Again, regard L as embedded in an associative algebra A, and consider the Lie subalgebra N of A generated by e_1^2 and e_2. From (2.3) one sees that N is an uncovered algebra of type 1, with N_1 spanned by e_1^2 and e_2 (where e_1^2 plays the role of e_1, and e_2 the role of y), and $N_i = \langle e_{2i} \rangle$ for $i \geq 2$. We say that N is obtained from L via **deflation.** Also, if $(\alpha_i)_{i\geq 3}$ is the sequence of two-step centralizers of L, then

$$[e_{2i} e_2] = \alpha_{2i} e_{2i+2} = \alpha_{2i} [e_{2i} e_1^2]$$

shows that the sequence $(\beta_i)_{i\geq 2}$ of two-step centralizers of N is given by $\beta_i = \alpha_{2i}$.

3. The length of the first constituent

In the rest of the paper all algebras are taken over a field $\mathbf{F}$ of characteristic 2.

Let L be an algebra of type 2. If the sequence of two-step centralizers has constant value α, then L is obtained via Lemma 2.3 from the unique metabelian algebra of type 1, whose sequence of two-step centralizers is constant [3, Lemma 3.1], for $\gamma = \alpha$.

We can thus assume from now on that the sequence of two-step centralizers is not constant.

Note also that if the first two-step centralizer α_3 is non-zero, we can apply Lemma 2.3 with $\gamma = \alpha_3$, and obtain an algebra with $\alpha_3 = 0$.

In this section we prove the following

3.1. PROPOSITION: *Let L be an algebra of type 2 with first two-step centralizer $\alpha_3 = 0$, whose sequence of two-step centralizers is not constant.*

Then its first constituent can have length of the form

1. *either $2q$, where $q > 2$ is a power of 2,*
1. *or $2q + 2$, where $q \geq 2$ is a power of 2.*

In the first case, after scaling the first constituent can be taken to be of the form $00\dots011$, that is, ordinary ending in 1. Here there are $2q - 4$ zeros before the 11.

In the second case, after scaling the first constituent can be taken to be of the form $00\dots010$, and it is thus not ordinary. Here there are $2q - 2$ zeros before the 10.

In Section 4 we prove, on the model of [5, Section 5], that if the first case of Proposition 3.1 occurs, then L occurs as a maximal subalgebra of an algebra of type 1.

In Section 5 we prove that if the second case of Proposition 3.1 occurs, then the sequence of two-step centralizers of L is obtained from that of a maximal subalgebra of an algebra of type 1 by adding $\gamma = 1$ to all terms of it, according to Lemma 2.3. This will prove Theorem 2.4.

Remark: In the first case of Proposition 3.1, we prefer to write $2q$ for the power of 2 that represents the length of the first constituent, because q corresponds then to the parameter, in the sense of Definition 2.5, of the underlying algebra of type 1.

In the second case of Proposition 3.1, q turns out to be a natural parameter for later usage. (See Section 5.)

Proof of Proposition 3.1: By assumption, not all two-step centralizers are zero. Scaling, we may assume that the first non-zero two-step centralizer is 1. So we consider sequences of two-step centralizers of the form $\underbrace{0\ldots0}_{k}1\ldots$. Using exponentiation to indicate repetition, we write $\underbrace{0\ldots0}_{k}1 = 0^k1$. Thus the first constituent of L has length $k+4$.

The first thing to note is that k must be even. This follows immediately from Lemma 2.1. Write $k = 2r$, so that $\alpha_i = 0$ for $3 \leq i \leq 2r+2$ and $\alpha_{2r+3} = 1$.

Using (2.2) of Lemma 2.1 for $m = r+2$, we get $\alpha_{2r+4} = r+1$. Since we are working in characteristic 2 we see that $\alpha_{2r+4} = 1$ if r is even, and that $\alpha_{2r+4} = 0$ if r is odd.

So if r is even, the sequence of two-step centralizers begins $0^{2r}11$. When we deflate L to an algebra N of type 1, the first constituent of N has length $r+2$. Because of Lemma 2.5, $r+2$ must be a power of 2, and the same holds for $k+4 = 2(r+2)$, so that we are in the first case of Proposition 3.1.

We consider now the case when r is odd, and show that this only arises when $2r = 2^m - 2$ for some $m \geq 2$. (The first constituent will thus have length $2^m + 2$, as claimed.) So we suppose that $2r = 2^m n - 2$ for some $m \geq 2$ and some odd $n \geq 1$. We show that this can only happen when $n = 1$. To this end we suppose that $n \geq 3$, and obtain a contradiction.

We pick t so that $2^{t-1} < 2^m n < 2^t$. Since $m \geq 2$ and $n \geq 3$ we see that $t \geq 4$. Since $2^{t-1}+1 \leq 2^m n = 2r+2$, $[e_{2^{t-1}+1}, e_2] = 0$. So if $3 \leq k \leq 2^{t-1}+1$ then

$$\begin{aligned}
0 &= [e_{2^m n+k-2^{t-1}-1}, [e_{2^{t-1}+1}, e_2]] \\
&= [e_{2^m n+k-2^{t-1}-1}, e_{2^{t-1}+1}, e_2] \\
&= [e_{2^m n+k-2^{t-1}-1}, [e_2, e_1^{2^{t-1}-1}], e_2] \\
&= \sum_{i=0}^{2^{t-1}-1} [e_{2^m n+k-2^{t-1}-1+i}, e_2, e_1^{2^{t-1}-1-i}, e_2] \\
&= \alpha_{2^m n+k} \left(\sum_{i=0}^{2^{t-1}-1} \alpha_{2^m n+k-2^{t-1}-1+i} \right) e_{2^m n+k+2}.
\end{aligned}$$

We want to use this equation to show that $\alpha_{2^m n+k} = 0$ for $3 \leq k \leq 2^{t-1}+1$. So we assume by induction that $\alpha_{2^m n+s} = 0$ for $3 \leq s < k$ (where $3 \leq k \leq 2^{t-1}+1$). This implies that

$$\sum_{i=0}^{2^{t-1}-1} \alpha_{2^m n+k-2^{t-1}-1+i} = 1,$$

and so $\alpha_{2^m n+k} = 0$, as required. But the equation $[e_{2^{t-1}+1}, e_{2^{t-1}+1}] = 0$ gives

$$0 = [e_{2^{t-1}+1}, e_{2^{t-1}+1}] = [e_{2^{t-1}+1}, [e_2, e_1^{2^{t-1}-1}]] = \left(\sum_{i=0}^{2^{t-1}-1} \alpha_{2^{t-1}+1+i} \right) e_{2^t+2},$$

which contradicts the results just proved.

So we can write $2r = 2^m - 2$ $(m \geq 2)$ as claimed, so that the sequence of two-step centralizers begins $0^{2^m-2}10$. ∎

4. First constituent of length $2q$, with $q > 2$

In this section we deal with the first case of Proposition 3.1, in which the first constituent has length $2q$, with $q = 2^h > 2$. This is the counterpart of [5, Section 5]: only a few changes are necessary to the arguments given there.

Suppose we have

$$[e_i e_2] = 0, \quad \text{for } i < 2q - 1;$$
$$[e_{2q-1} e_2] = e_{2q+1}, \quad [e_{2q} e_2] = e_{2q+2}.$$

We want to show that the algebra is a maximal subalgebra of an algebra of type 1, by proving that all constituents are ordinary, and invoking Proposition 2.2.

Proceeding by induction, assume we have already proved this up to a certain constituent, that ends as

$$[e_m e_2] = \lambda e_{m+2}, \quad [e_{m+1} e_2] = \lambda e_{m+3}, \tag{4.1}$$

for some $\lambda \neq 0$.

The argument of [5, Section 5] carries over verbatim to show that $2q$ is an upper bound for the length of the next constituent.

We now show that q is a lower bound for the length of the next constituent, that is

$$[e_{m+2} e_2] = [e_{m+3} e_2] = \cdots = [e_{m+q-1} e_2] = 0.$$

As in [5, Section 5], we do this more generally for the case when the current constituent is of the general form

$$[e_m e_2] = \mu e_{m+2}, \quad [e_{m+1} e_2] = \nu e_{m+3}, \tag{4.2}$$

where $\mu \neq 0$, but where we make no assumption on ν. We proceed by induction, so that we can assume the current constituent to have length at least q.

The argument of [5, Section 5] for the case $\nu = 0$ in (4.2) carries over here. Suppose thus $\nu \neq 0$.

We show that $[e_{m+l+2}e_2] = 0$ for $0 \leq l < q-2$, proceeding by induction on l. We have first

$$0 = [e_{m-1}[e_2e_1e_2]] = \mu[e_{m+2}e_2],$$

so that $[e_{m+2}e_2] = 0$. Suppose thus $l > 0$. We compute

$$\begin{aligned} 0 &= [e_m[e_2e_1^l e_2]] \\ &= [e_m[e_2e_1^l]e_2] - [e_m e_2[e_2e_1^l]] \\ &= (\mu + l\nu + \mu)[e_{m+l+2}e_2] \\ &= l\nu[e_{m+l+2}e_2]. \end{aligned}$$

The coefficient $l\nu$ vanishes when l is even. In this case, write $l = 2^t\beta$, with β odd. If $t > 1$, note that $l = 2^t\beta < q-2$, and thus $2^t < q$. Therefore $l + 2^t < 2q-3$, and $[e_{l+2^t}e_2] = 0$. Thus we have

$$\begin{aligned} 0 &= [e_{m-2^t+2}[e_2e_1^{l+2^t-2}e_2]] \\ &= [e_{m-2^t+2}[e_2e_1^{l+2^t-2}]e_2] \\ &= \left(\mu\binom{l+2^t-2}{2^t-2} + \nu\binom{l+2^t-2}{2^t-1}\right)\cdot[e_{m+l+2}e_2] \\ &= \mu[e_{m+l+2}e_2]. \end{aligned}$$

If $t = 1$, that is, $l = 2\beta$, choose $n \geq 2$ so that

$$\beta \equiv 2^{n-1} - 1 \pmod{2^n},$$

and let $k = 2^n - 1$. Since $l = 2\beta < q-2$, we have $\beta < q/2 - 1$, and thus $2^{n-1} < q/2$, or $2^n < q$. It follows that $l + k < 2q-3$, so that $[e_2e_1^{l+k}e_2] = 0$. We may now compute

$$\begin{aligned} 0 &= [e_{m-k}[e_2e_1^{l+k}e_2]] \\ &= \left(\mu\binom{2\beta+k}{k} + \nu\binom{2\beta+k}{k+1}\right)\cdot[e_{m+l+2}e_2] \\ &= \nu[e_{m+l+2}e_2]. \end{aligned}$$

This is because we have, for some γ,

$$\begin{aligned} \binom{2\beta+k}{k} &\equiv \binom{\gamma 2^{n+1} + 2^n - 2 + 2^n - 1}{2^n - 1} \\ &\equiv \binom{\gamma 2^{n+1} + 2^n + 2^{n-1} + \ldots + 4 + 1}{2^{n-1} + \ldots + 4 + 2 + 1} \\ &\equiv 0 \pmod{2}, \end{aligned}$$

while

$$\begin{aligned}\binom{2\beta+k}{k+1} &\equiv \binom{\gamma 2^{n+1}+2^n+2^{n-1}+\ldots+4+1}{2^n}\\ &\equiv 1 \pmod 2.\end{aligned}$$

This proves that q is a lower bound for the length of the next constituent. From now on, the rest of the argument of [5] applies.

5. First constituent of length $2q+2$, with $q \geq 2$

In this section we deal with the second case of Proposition 3.1. So assume L is an algebra of type 2, with first constituent of length $2q+2$, where $q \geq 2$ is a power of 2. So its sequence of two-step centralizers begins $0^{2q-2}10$.

We first show that the sequence is followed by another zero. In fact, if the sequence continues as $0^{2q-2}10\alpha_{2q+3}$, we have

$$\begin{aligned} 0 &= [e_{2q+1}[e_1e_2e_2]]\\ &= [e_{2q+1}e_1e_2e_2] + [e_{2q+1}e_2e_2e_1]\\ &= (\alpha_{2q+2}\alpha_{2q+4} + \alpha_{2q+1}\alpha_{2q+3}) \cdot e_{2q+6}\\ &= \alpha_{2q+3}e_{2q+6},\end{aligned}$$

since $\alpha_{2q+1} = 1$ and $\alpha_{2q+2} = 0$, so that $\alpha_{2q+3} = 0$.

We now apply Lemma 2.3 with $\gamma = 1$. The resulting algebra, that we rename L, will have a sequence of two-step centralizers starting as $1^{2q-2}011$, so that all constituents so far are ordinary ending in 1, and the sequence of their lengths begin as $42^{q-2}3$. We prove

5.1. PROPOSITION: *Let L be an algebra of type 2. Suppose its first constituents are ordinary ending in 1, and that their lengths are $42^{q-2}3$, where $q > 1$ is a power of 2.*

Then all constituents are ordinary ending in 1, and the sequence of their lengths continues with repetitions of the pattern $2^{q-2}3$ or $2^{q-2}4$.

We need only the statement about all constituents being ordinary in order to be able to apply Proposition 2.2. The statement about the sequence of the lengths of the constituents, however, is an essential ingredient in our proof by induction.

The algebras that are covered by Proposition 5.1 are maximal subalgebras of

- the algebras $\mathrm{AFS}(1, b, n, 2)$ of Shalev ([12, 3]; we use the notation of [4], so that $1 < b \leq n$, $q = 2^{b-1}$, and n may also be infinity), and
- the Bi-Zassenhaus algebras of [9, 10].

Proof of Proposition 5.1: We will use some identities. The first one comes from $\alpha_3 = 1$, that is, $[e_2e_1e_2] = [e_2e_1^3]$. We compute

$$\begin{aligned} 0 &= [e_n[e_1e_2e_2]] + [e_n[e_2e_1^3]] \\ &= [e_ne_1e_2e_2] + [e_ne_2e_2e_1] + [e_ne_2e_1^3] + [e_ne_1e_2e_1^2] + [e_ne_1^2e_2e_1] + [e_ne_1^3e_2] \\ &= (\alpha_{n+1}\alpha_{n+3} + \alpha_n\alpha_{n+2} + \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3}) \cdot e_{n+5}. \end{aligned}$$

That is, for $n \geq 3$ we have

$$\alpha_n\alpha_{n+2} + \alpha_{n+1}\alpha_{n+3} + \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3} = 0. \tag{5.1}$$

The second identity is based on $\alpha_{2q+1} = 0$, that is, $[e_2e_1^{2q-1}e_2] = 0$. We compute

$$0 = [e_n[e_2e_1^{2q-1}e_2]] = [e_ne_2[e_2e_1^{2q-1}]] + [e_n[e_2e_1^{2q-1}]e_2],$$

and obtain

$$\begin{aligned} &\alpha_n \cdot (\alpha_{n+2} + \cdots + \alpha_{n+2q} + \alpha_{n+2q+1}) \\ &\qquad + \alpha_{n+2q+1} \cdot (\alpha_n + \alpha_{n+1} + \ldots + \alpha_{n+2q-1}) = 0. \end{aligned} \tag{5.2}$$

Finally, we have a string of identities, encoding $\alpha_5 = \cdots = \alpha_{2q-1} = 1$. (There are no such identities when $q = 2$.) For k odd, $5 \leq k \leq 2q - 1$, we have $[e_2e_1^{k-2}e_2] = [e_2e_1^k]$. We compute

$$0 = [e_ne_2[e_2e_1^{k-2}]] + [e_n[e_2e_1^{k-2}]e_2] + [e_n[e_2e_1^k]],$$

and obtain

$$\begin{aligned} &\alpha_n \cdot \left(\sum_{i=0}^{k-2} \binom{k-2}{i} \alpha_{n+2+i}\right) + \alpha_{n+k} \cdot \left(\sum_{i=0}^{k-2} \binom{k-2}{i} \alpha_{n+i}\right) \\ &\qquad + \sum_{i=0}^{k} \binom{k}{i} \alpha_{n+i} = 0. \end{aligned} \tag{5.3}$$

When applying one of these identities, we will underline on the sequence of two-step centralizers the first and the last one involved. For instance, when using (5.1) we will underline α_n and α_{n+3}, or also say we are applying the identity on $\alpha_n\alpha_{n+1}\alpha_{n+2}\alpha_{n+3}$.

Proceeding by induction, we first assume we know the sequence of lengths of two-step centralizers up to a segment of the form $2^{q-2}4$, with all constituents ordinary ending in 1, so far.

The tail of the sequence of two-step centralizers is thus of the form

$$011 \mid 11 \mid \cdots \mid 11 \mid 0011,$$

where there are $q-2$ repetitions of the pattern 11. (The leading 011 might well be a piece of a 0011.) We want to show first, using secondary induction, that there are $q-2$ further repetitions of the pattern 11 after 011. (Clearly, this is not needed when $q=2$.)

So we assume we have

$$011 \mid 11 \mid \cdots \mid 11 \mid 0011 \mid 11 \mid \cdots \mid 11 \mid \alpha\beta,$$

with less than $q-2$ instances of 11 between 0011 and $\alpha\beta$, and we want to show $\alpha=\beta=1$.

For later usage, we deal with the slightly more general

$$011 \mid 11 \mid \cdots \mid 11 \mid 0\omega 11 \mid 11 \mid \cdots \mid 11 \mid \alpha\beta,$$

where $\omega \neq 1$.

We first use (5.2) on a suitable

$$1\underline{1} \mid 11 \mid \cdots \mid 11 \mid 0\omega 11 \mid 11 \mid \cdots \mid 11 \mid \underline{\alpha}\beta,$$

to get $1(1+\omega+\alpha)+\alpha(1+\omega+1)=(\omega+1)(\alpha+1)=0$, so that $\alpha=1$.

We then use again (5.2) on

$$\underline{1}1 \mid 11 \mid \cdots \mid 11 \mid 0\omega 11 \mid 11 \mid \cdots \mid 11 \mid 1\underline{\beta},$$

to get $1(1+\omega+\beta)+\beta(\omega)=(\omega+1)(\beta+1)=0$, so that $\beta=1$.

So we are at

$$011 \mid 11 \mid \cdots \mid 11 \mid 0011 \mid 11 \mid \cdots \mid 11 \mid \alpha\beta\gamma\delta,$$

with $q-2$ instances of 11 between 0011 and α. (We have gone back to $\omega=0$.)

We first use (5.2) on

$$011 \mid 11 \mid \cdots \mid 1\underline{1} \mid 0011 \mid 11 \mid \cdots \mid 11 \mid \underline{\alpha}\beta\gamma\delta$$

to get $\alpha=0$.

Now we use (5.1) on $10\beta\gamma$ to get $\gamma=1$. So if $\beta=1$ we have a constituent of length three, and we are done. Suppose thus $\beta\neq 1$.

Then we use (5.1) again on $0\beta 1\delta$ to get $\beta\delta+1+\beta+\delta=(\beta+1)(\delta+1)=0$. Since $\beta\neq 1$, we have $\delta=1$. If $\beta=0$, we now have a constituent of length four, and we are done. So suppose $\beta\notin\{0,1\}$. We are now at

$$011 \mid 11 \mid \cdots \mid 11 \mid 0011 \mid 11 \mid \cdots \mid 11 \mid 0\beta 11.$$

Since $\beta \neq 1$, the argument above allows us to add $q-2$ more instances of 11 to get

$$011 \mid 11 \mid \cdots \mid 11 \mid 0011 \mid 11 \mid \cdots \mid 11 \mid 0\beta 11 \mid 11 \mid \cdots \mid 11 \mid \sigma. \tag{5.4}$$

We use (5.2) on

$$1\underline{1} \mid 0\beta 11 \mid 11 \mid \cdots \mid 11 \mid \underline{\sigma}$$

to get

$$\beta + \sigma + \beta\sigma = 0. \tag{5.5}$$

Thus $\sigma \notin \{0, 1\}$. Because of Lemma 2.1, β occurs as an odd indexed two-step centralizer α_i. Therefore σ occurs with an even index. Recall that the leading 011 might be a full constituent of length three, or part of one 0011 of length four.

Since the lengths of the constituents of L begin $42^{q-2}3$, there is at least one constituent of length 3 in the sequence leading up to β.

First consider the case when there is only one constituent of length three in the sequence leading up to β, and all other constituents have length 2 or 4.

Deflation and Lemma 2.8 yield that σ appears with an index of the form $2r$, where $r > 1$ is a power of 2.

Applying (2.2) for $m = r$ we see

$$\sigma = \alpha_{2r} = \sum_{i=0}^{r-2} (-1)^{r+i} \binom{r-1}{i} \alpha_{r+1+i} = \sum_{i=0}^{r-2} \alpha_{r+1+i}. \tag{5.6}$$

Now β occurs once in the summation. On the other hand, the ones occur in pairs between α_{r+1} and α_{2r-1}, because all constituents are of length two or four, so they cancel out. (One sees easily that α_{r+1} is the second 0 in a 0011 constituent of length four, so we are starting with the right offset.)

We obtain from (5.6) $\sigma = \beta$, so that (5.5) yields $\beta = 0$, a contradiction.

Next suppose that there are at least two constituents of length 3 in the sequence leading up to β. We first deflate the sequence (5.4) to get an algebra N of type 1 with a segment of the sequence of two-step centralizers of the form

$$\ldots 1^{q-1} 0 1^{q-1} \sigma.$$

Because of Lemmas 2.5 and 2.7, the parameter of N is q, its first two-step centralizer in order of occurrence is 1, the second one 0, and the third one $\sigma \notin \{0, 1\}$.

If the leading 011 in (5.4) is a full constituent of length three, we can extend the sequence backwards with $q-2$ more instances of 11 to

$$011 \mid 11 \mid \cdots \mid 11 \mid 011 \mid 11 \mid \cdots \mid 11 \mid 0011 \mid 11 \mid \cdots \mid 11 \mid 0\beta 11 \mid 11 \mid \cdots \mid 11 \mid \sigma.$$

(Note that we have a leading zero in this sequence because we are assuming that there are at least two constituents of length 3 in the sequence leading up to β.) Inspection now reveals a constituent of intermediate length $2q-1$ in the deflated algebra. This contradicts Lemma 2.6. This argument can be easily extended to show that we cannot have more than one constituent of length three in the sequence leading up to β, wherever they may occur.

We now assume we know the sequence of lengths of two-step centralizers up to a segment of the form $2^{q-2}3$, with all constituents ordinary ending in 1, so far.

Proceeding as in the previous case, we assume the tail of the sequence of two-step centralizers has the form

$$011 \mid 11 \mid \cdots \mid 11 \mid 011,$$

where there are $q-2$ repetitions of the pattern 11, and show first that there are $q-2$ further repetitions of the pattern 11 after 011. (There is a special case, when we are at the very beginning of the sequence of two-step centralizers, so the above sequence is just $11 \mid 11 \mid \cdots \mid 11 \mid 011$, with $q-1$ repetitions of the pattern 11. We will deal with this at the end of the proof.)

Proceeding by secondary induction, we assume we have already

$$011 \mid 11 \mid \cdots \mid 11 \mid 011 \mid 11 \mid \cdots \mid 11 \mid \alpha\beta,$$

with less than $q-2$ repetitions of 11 between 011 and $\alpha\beta$, and we want to show $\alpha = \beta = 1$.

We first apply (5.2) to a suitable

$$1\underline{1} \mid 11 \mid \cdots \mid 11 \mid 011 \mid 11 \mid \ldots \mid 11 \mid \alpha\underline{\beta},$$

to get $1(1+\alpha+\beta)+\beta(1) = \alpha+1 = 0$, where we have omitted pairs of 1 that cancel out. So $\alpha = 1$.

Now we apply a suitable instance of (5.3) to

$$1\underline{1} \mid 011 \mid 11 \mid \cdots \mid 11 \mid 1\underline{\beta},$$

to get $1(1+\beta)+\beta(1+0+1+1)+1+0+1+\beta = \beta+1 = 0$, so that $\beta = 1$. A word of comment is in order here. Since k is odd in (5.3), we have

$$\binom{k}{0} \equiv \binom{k}{1} \equiv \binom{k}{k-1} \equiv \binom{k}{k} \equiv 1 \pmod 2,$$

and similarly for binomial coefficients involving $k-2$. We do not need any information about the other binomial coefficients, because they multiply α's that are 1, and thus cancel out in pairs, such as

$$\binom{k}{2}\alpha_{n+2}+\binom{k}{k-2}\alpha_{n+k-2}=0.$$

Once all the $q-2$ instances of 11 are in place, we suppose we have

$$011 \mid 11 \mid \cdots \mid 11 \mid 011 \mid 11 \mid \cdots \mid 11 \mid \alpha\beta\gamma\delta.$$

We use (5.2) on

$$011 \mid 11 \mid \cdots \mid 1\underline{1} \mid 011 \mid 11 \mid \cdots \mid 11 \mid \alpha\underline{\beta}\gamma\delta$$

to get $\alpha = 0$.

Now we use (5.1) on $10\beta\gamma$ to get $\beta+1+0+\beta+\gamma=\gamma+1=0$, that is $\gamma=1$.

Note that $\beta=1$ would give us a constituent 011 of length three, and we are done, so we may take $\beta \neq 1$.

Now we use (5.1) on $0\beta1\delta$ to get $\beta\delta+1+\beta+\delta=(\beta+1)(\delta+1)=0$. Since $\beta \neq 1$, we have $\delta = 1$.

Note that $\beta = 0$ would give us a constituent 0011 of length four, and we are done. So we may assume $\beta \notin \{0,1\}$.

The argument used for the previous case allows us to extend this to

$$011 \mid 11 \mid \cdots \mid 11 \mid 0\beta 11 \mid 11 \mid \cdots \mid 11 \mid \sigma,$$

where there are $q-2$ instances of 11 between $0\beta11$ and σ.

As above, we use (5.2) on

$$1\underline{1} \mid 0\beta 11 \mid 11 \mid \cdots \mid 11 \mid \underline{\sigma}$$

to get $\beta+\sigma+\beta\sigma=0$. So $\sigma \notin \{0,1\}$. Because of Lemma 2.1, β occurs as an odd indexed two-step centralizer α_i. Therefore σ occurs with an even index. We now deflate the sequence

$$011 \mid 11 \mid \cdots \mid 11 \mid 011 \mid 11 \mid \cdots \mid 11 \mid 0\beta 11 \mid 11 \mid \cdots \mid 11 \mid \sigma,$$

to get an algebra N of type 1 with a segment of the sequence of two-step centralizers of the form

$$01^{2q-2}01^{q-1}\sigma.$$

As above, in N the parameter is q, the first two-step centralizer in order of occurrence is 1, the second one is 0, and the third one is $\sigma \notin \{0,1\}$. Also, the $1^{2q-2}0$ gives a constituent of intermediate length $2q-1$. The fact that $\sigma \notin \{0,1\}$ now contradicts Lemma 2.6.

We are left with the special case when we are at the very beginning of the sequence of two-step centralizers. In this case the above arguments yield that the sequence begins

$$11 \mid 11 \mid \cdots \mid 11 \mid 011 \mid 11 \mid \cdots \mid 11 \mid 0\beta 11 \mid 11 \mid \cdots \mid 11 \mid \sigma.$$

Simple counting shows that σ occurs as α_{6q} in the sequence of two-step centralizers. Deflation and Lemma 2.8 yield that $3q$ should be a power of 2. This contradiction completes the proof. ∎

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